9 May 2010

Baez + Munian exercise 10–14

10. $v = w \iff v(f) = w(f)$ for all $f \in C^\infty(M)$

$\iff v(f)(p) = w(f)(p)$ for all $f \in C^\infty(M)$, $p \in M$

i.e. $v_p = w_p$ for all $p \in M$.

11. All the vector space axioms hold reasonably manifestly.

We should check that if $v, w \in T_pM$ then $v+w$ is indeed in T_pM. Must check Leibniz:

$$(v+w)(fg) = v(fg) + w(fg) \qquad [\text{defn}]$$

$$= v(f)g(p) + f(p)v(g) + w(f)g(p) + f(p)w(g) \qquad [v, w \in T_pM]$$

$$= v(f)g(p) + w(f)g(p) + f(p)v(g) + f(p)w(g) \qquad [\text{commutativity of addition}]$$

$$= (v+w)(f)\,g(p) + f(p)\,(v+w)(g). \qquad [\text{defn}]$$

12. $$\gamma'(t)(f+g) = \frac{d}{dt}(f+g)(\gamma(t)) \qquad [\text{defn of } \gamma'(t)]$$

$$= \frac{d}{dt}\left[f(\gamma(t)) + g(\gamma(t))\right] \qquad [\text{defn of } f+g]$$

$$= \frac{d}{dt} f(\gamma(t)) + \frac{d}{dt} g(\gamma(t)) \qquad [\text{derivative of sum} = \text{sum of derivatives for functions } \mathbb{R} \to \mathbb{R}]$$

$$= \gamma'(t)(f) + \gamma'(t)(g).$$

$$\gamma'(t)(\alpha f) = \frac{d}{dt}(\alpha f)(\gamma(t)) \qquad [\text{defn of } \gamma'(t)]$$

$$= \frac{d}{dt}\left[\alpha f(\gamma(t))\right] \qquad [\text{defn of } \alpha f]$$

$$= \alpha \frac{d}{dt} f(\gamma(t))$$

[derivative of a constant times a function = the constant times the derivative of the function for functions $\mathbb{R} \to \mathbb{R}$].

$$\gamma'(t)(fg) = \frac{d}{dt}(fg)(\gamma(t)) \qquad [\text{defn of } \gamma'(t)]$$

$$= \frac{d}{dt}\left[f(\gamma(t))\, g(\gamma(t))\right] \qquad [\text{defn of } f.g]$$

$$= \left[\frac{d}{dt} f(\gamma(t))\right] g(\gamma(t)) + f(\gamma(t)) \frac{d}{dt} g(\gamma(t))$$

[ordinary product rule for functions $\mathbb{R} \to \mathbb{R}$]

$$= \gamma'(t)(f)\, g(\gamma(t)) + f(\gamma(t))\, \gamma'(t)(g). \qquad [\text{defn of } \gamma'(t)].$$

13.
$$(\phi^* x)(t) = (x \circ \phi)(t) \qquad [\text{defn}]$$

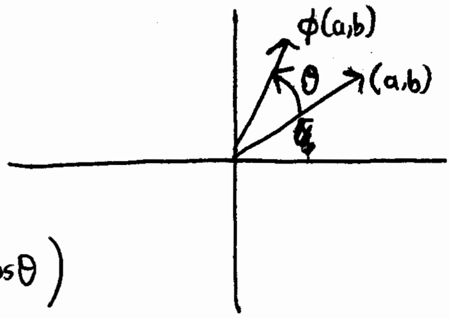
$$= x(e^t) = e^t$$

while
$$e^x(t) = e^{x(t)} \qquad [\text{defn}]$$

$$= e^t \qquad [\text{defn of } x].$$

14. Clearly we have

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(a,b) \longmapsto (a\cos\theta - b\sin\theta,\ a\sin\theta + b\cos\theta)$$



So
$$\begin{aligned}
[\phi^* x](a,b) &= [x \circ \phi](a,b) && (\text{defn of } \phi^*) \\
&= x(a\cos\theta - b\sin\theta,\ a\sin\theta + b\cos\theta) && (\text{defn of } \phi) \\
&= a\cos\theta - b\sin\theta && (\text{defn of } x)
\end{aligned}$$

while
$$\begin{aligned}
[(\cos\theta)x - (\sin\theta)y](a,b) &= (\cos\theta)x(a,b) - (\sin\theta)y(a,b) && [\text{defn of linear combination of functions}] \\
&= \cos\theta \cdot a - \sin\theta \cdot b.
\end{aligned}$$

Similarly,
$$\begin{aligned}
[\phi^* y](a,b) &= [y \circ \phi](a,b) \\
&= y(a\cos\theta - b\sin\theta,\ a\sin\theta + b\cos\theta) \\
&= a\sin\theta + b\cos\theta
\end{aligned}$$

while
$$\begin{aligned}
[(\sin\theta)x + (\cos\theta)y](a,b) &= (\sin\theta)x(a,b) + (\cos\theta)y(a,b) \\
&= \sin\theta \cdot a + \cos\theta \cdot b
\end{aligned}$$