Exercise 50

Let the spacetime be any manifold $\mathcal{M}$ of any dimension.
We can write $\mathcal{M}$ as ~~the~~ product of $\mathbb{R}$ and S, that is

$$\mathcal{M} = \mathbb{R} \times S \qquad (1)$$

where the manifold S is for space and $\mathbb{R}$ is for time.

Also $\dim \mathcal{M} = \dim \mathbb{R} + \dim S \qquad (2)$

Let θ^α be ~~the local~~ coordinates on $\mathbb{R} \times S$, then if F is a 2-form on $\mathbb{R} \times S$ we have:

$$F = \frac{1}{2} \sum_{\alpha,\beta \in \{0,1,\cdots n\}} F_{\alpha\beta}\, d\theta^\alpha \wedge d\theta^\beta$$

$$= \frac{1}{2} \sum_{\beta \in \{1,\cdots,n\}} F_{0\beta}\, d\theta^0 \wedge d\theta^\beta + \frac{1}{2} \sum_{\alpha \in \{1,\cdots n\}} F_{\alpha 0}\, d\theta^\alpha \wedge d\theta^0$$

$$+ \frac{1}{2} \sum_{\alpha,\beta \in \{1,\cdots n\}} F_{\alpha\beta}\, d\theta^\alpha \wedge d\theta^\beta$$

$$F = \sum_{\alpha \in \{1 \cdots n\}} F_{\alpha 0}\, d\theta^\alpha \wedge d\theta^0 + \frac{1}{2} \sum_{\alpha,\beta \in \{1,\cdots,n\}} F_{\alpha\beta}\, d\theta^\alpha \wedge d\theta^\beta \qquad (3)$$

where we have used the fact that
$F_{0\beta} = -F_{\beta 0}$ and $d\theta^0 \wedge d\theta^\beta = -d\theta^\beta \wedge d\theta^0$

If we make the following substitution:
$F_{\alpha 0} = E_i$, $F_{\alpha\beta} = B_{ij}$, $d\theta^0 = dt$, $d\theta^\alpha = dx^i$
and $d\theta^\beta = dx^j$

Therefore
$$F = B + E \wedge dt \qquad (4)$$
where $B = \frac{1}{2} B_{ij}\, dx^i \wedge dx^j$ and $E = E_i\, dx^i$
which shows that any ~~two~~ 2-form on
$\mathbb{R} \times S$ can be expressed as (4)
such that if x^i is any local coordinate
of S we have
$E_j = E_i\, dx^i$ and $B = \frac{1}{2} B_{ij}\, dx^i \wedge dx^j$.