Exercise 69

Solomon

May 3, 2010

Define the Levi-Civita symbol by

$$\epsilon_{i_1,\ldots,i_n} = \begin{cases} \text{sign}\,(i_1,\ldots,i_n) & \text{all } i_j \text{ distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

or

$$\epsilon_{i_1,\ldots,i_p} = \begin{cases} +1 & \text{if } (i_1,\ldots,i_p) \text{ is an even permutation of } (1,\ldots,n), \\ -1 & \text{if } (i_1,\ldots,i_p) \text{ is an odd permutation of } (1,\ldots,n), \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\omega = \frac{1}{p!}\omega_{i_1\cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p}$$

Taking the dual of both sides, we obtain:

$$\begin{aligned} \star\omega &= \frac{1}{p!}\omega_{i_1\cdots i_p} \star \left(e^{i_1} \wedge \cdots \wedge e^{i_p}\right) \\ &= \frac{1}{p!}\text{sign}\,(i_1,\ldots,i_n)\,\epsilon(i_1)\cdots\epsilon(i_p)\omega_{i_1\cdots i_p}\, e^{j_1} \wedge \cdots \wedge e^{j_{n-p}} \\ &= \frac{1}{p!(n-p)!}\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}\omega_{i_1\cdots i_p} e^{j_1} \wedge \cdots \wedge e^{j_{n-p}} \\ &= \frac{1}{(n-p)!} \star(\omega)_{j_1\cdots j_{n-p}} e^{j_1} \wedge \cdots \wedge e^{j_{n-p}} \end{aligned}$$

where, $\star(\omega)_{j_1\cdots j_{n-p}} = \frac{1}{p!}\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}\omega_{i_1\cdots i_p}$ as required.

Notice that we have used the linear property of Hodge star operator and the sign $(i_1,\ldots,i_n)\,\epsilon(i_1)\cdots\epsilon(i_p)$ is being replaced by the Levi-Civita symbol $\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}$, where, $j_1 <,\ldots,< j_{n-p}$ is the complement of $i_1 <,\ldots,< i_p$ in the set $\{1,\ldots,n\}$. The $(n-p)!$ takes care of double counting due to the antisymmetric permutation symbol.